

# The simple non-degenerate relativistic gas: statistical properties and Brownian motion

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## Abstract

This paper shows a novel calculation of the mean square displacement of a classical Brownian particle in a relativistic thermal bath. The result is compared with the expressions obtained by other authors. Also, the thermodynamic properties of a non-degenerate simple relativistic gas are reviewed in terms of a treatment performed in velocity space.

## 1 Introduction

Interest in relativistic statistical mechanics has been revived due to its applicability in several fields of study and the need of its internal theoretical consistence [1][2]. New experimental devices, such as relativistic heavy ion colliders, have also drawn attention to this topic since they provide a new framework to test different approaches to relativistic transport theory [3]. In the context of stochastic processes theory, formalisms involving relativistic thermodynamics have been proposed in order to understand the nature of how relativity may broaden the usual conceptions of mathematical probability theory.

In this sense, the analysis of processes such as Brownian motion in a relativistic framework deserves a deeper look. Other issues related to the inclusion of non-equilibrium effects in relativity theory have been addressed in previous work [4][5]. In the case of large (relativistic) temperatures, a description involving a Juttner function is required in order to compute the mean square displacement of Brownian particles [6]. This calculation is described in the present work. The interest of Brownian motion has been recently revived due to its applicability in different areas, including astrophysics [9].

This paper is divided as follows. In section two, the basic properties of a non-degenerate relativistic gas are reviewed and compared with their Maxwell-Boltzmann function counterparts at mildly and ultra-relativistic regimes. Section three is devoted to the study of relativistic Brownian

motion modeling the dynamics of a Newtonian brownian particle in a relativistic thermal bath. Due to this approximation, only elementary mathematical tools are involved in the calculations. Finally, some theoretical remarks on the use of relativistic statistical mechanics are included in section four.

## 2 Distribution functions

Some thermodynamic systems cannot be modeled using the usual Maxwell-Boltzmann distribution. Instead, a more appropriate equilibrium distribution function should be considered in high temperature cases. For example, the Jüttner distribution function for a special relativistic non-degenerate simple gas at rest is [1]:

$$f(\gamma; z) = \frac{e^{-\frac{\gamma}{z}}}{4\pi z c^3 K_2\left(\frac{1}{z}\right)}, \quad (1)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{w^2}{c^2}}}, \quad (2)$$

is the usual Lorentz factor for molecular speed  $w$ ,  $z = \frac{kT}{mc^2}$  is the relativistic parameter at temperature  $T$  for the system consisting of particles of mass  $m$ ,  $K_2$  is the modified Bessel function of the second kind,  $k$  and  $c$  are Boltzmann's constant and the speed of light, respectively.

Figure 1 shows a comparison between both speed distribution functions as functions of  $\beta = \frac{w}{c}$ , corresponding to Eq. (1) and a Maxwell-Boltzmann distribution written in terms of  $\gamma$ :

$$f_{MB}(\gamma; z) = \left(\frac{1}{2\pi z c^2}\right)^{3/2} \frac{4\pi c^2(\gamma^2 - 1)}{\gamma^2} e^{-\frac{\gamma^2 - 1}{2z\gamma^2}} = n \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}}, \quad (3)$$

for an electron gas. Although the differences seem small, they are measurable and can be detected in certain astrophysical processes such as the Sunyaev-Zel'dovich effect [7].

In special relativity, the four-vector  $v^\nu$  is related to the three velocity  $\vec{w}$  by:

$$v^\nu = \begin{pmatrix} \gamma \vec{w} \\ \gamma c \end{pmatrix}, \quad (4)$$

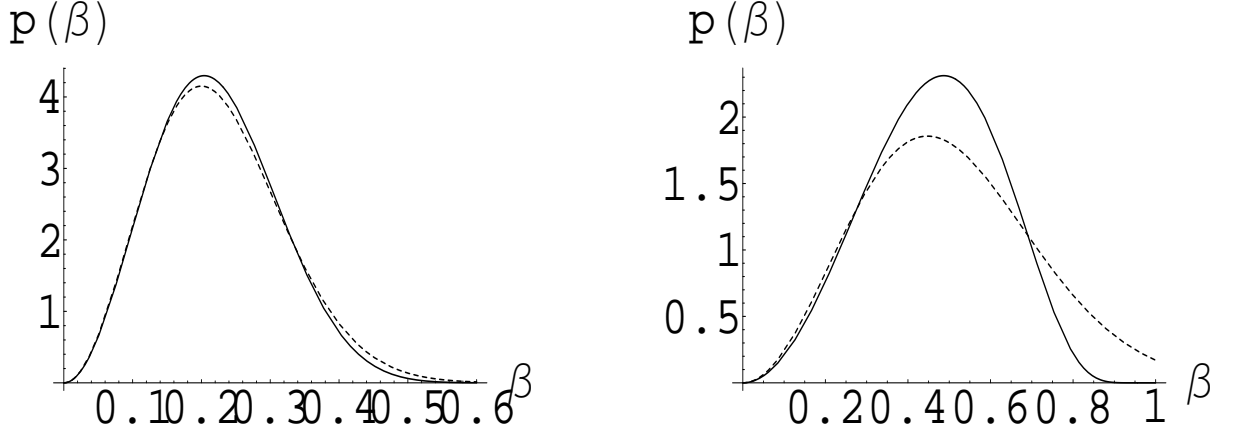
and the volume element in velocity space reads  $d^3v = 4\pi w^2 \gamma^5 dw$  [8]. One fundamental difference between both distributions, Maxwellian and Jüttnerian, is that the former allows for molecular speeds greater than  $c$ , while the latter has a cutoff at  $\beta = 1$ . Both behaviors are shown in Fig. 1. On the first plot both distributions have a similar qualitative behavior, where most particles lie on the right side of the peaks. However, for larger values of  $z$ , as can be seen in the second plot, both distributions are quite different. Notice how the dotted, classical, distribution exceeds the physical limit  $\beta = 1$  while the Jüttner distribution function vanishes exactly at that value.

We are now in position to address a simple application of relativistic thermodynamics and the corresponding comparison with the non-relativistic case.

## 3 A simple approach to relativistic Brownian motion

Before addressing the calculation of the mean internal energy per particle for a non-degenerate gas, both in the non-relativistic and relativistic cases. In the non-relativistic case, this quantity is identified with the average integral:

Figure 1: This figure shows the probability distribution functions, for  $z = 0.01$  (left) and  $z = 0.1$  (right), both in the classical case (dotted line) and in the relativistic case (solid line). The small differences at  $z = 0.01$ , corresponding to an electron temperature of  $6 \times 10^7 K$  for an electron gas, are large enough to account for relativistic corrections to the Sunyaev-Zel'dovich effect in clusters of galaxies. The relativistic features of the probability function are significantly enhanced for  $z = 0.1$ , corresponding to an electron temperature of  $6 \times 10^8 K$ .



$$\langle \epsilon \rangle = \int_0^\infty \frac{1}{2} m v^2 f_{MB}(v) dv = \frac{1}{2} m^{\frac{5}{2}} \left( \frac{1}{2\pi z} \right)^{3/2} \int_0^\infty v^4 e^{-\frac{m v^2}{2kT}} dv. \quad (5)$$

The usual change of variable  $\omega = \sqrt{\frac{m}{2kT}} v$  leads to the result:

$$\langle \epsilon \rangle = \frac{4}{\sqrt{\pi}} kT \int_0^\infty \omega^4 e^{-\omega^2} d\omega = \frac{3kT}{2}. \quad (6)$$

This result is identified with the energy equipartition theorem. Each degree of freedom contributes  $\frac{kT}{2}$  units of energy to the total internal energy per particle of the system. As we shall see below, the energy equipartition theorem does not hold in relativistic thermodynamics.

In the relativistic case, since the kinetic energy per particle is  $mc^2\gamma$ , the counterpart of Eq.(5) reads:

$$\langle \epsilon \rangle = \int_0^c mc^2 \gamma f(w) (4\pi w^2 \gamma^5) dw. \quad (7)$$

In terms of  $\gamma$  Eq. (7) reads:

$$\langle \epsilon \rangle = \frac{mc^2}{z K_2\left(\frac{1}{z}\right)} \int_1^\infty \gamma^2 e^{-\frac{\gamma}{z}} (\gamma^2 - 1)^{\frac{1}{2}} d\gamma, \quad (8)$$

which yields an expression for  $\langle \epsilon \rangle$  in terms of modified Bessel functions of the second kind,

namely [1]

$$\langle \epsilon \rangle = 3kT + mc^2 \frac{K_1\left(\frac{1}{z}\right)}{K_2\left(\frac{1}{z}\right)}. \quad (9)$$

Equation (9) reflects the fact that, for low enough temperatures, internal energy is well described by a linear function of the temperature  $T$ , and a non-linear behavior is exhibited for increasing  $z$ . The use of asymptotic expansions for the modified Bessel functions present in this equation, for  $z \ll 1$  leads to the result of Eq. (6) [10].

In order to account for the mean square displacement  $\langle r^2 \rangle$  of a *classical* Brownian particle of mass  $M$  in a relativistic gas, we assume that the relativistic parameter  $z$  of the gas is not negligible, while the equation of motion describing the displacement still reads:

$$M \frac{d\vec{v}}{dt} + \mu \vec{v} = \vec{F}_a, \quad (10)$$

where  $F_a$  is a random force,  $\mu$  is the dynamic friction coefficient and  $\vec{v}$  is the velocity of the particle. As usual, we assume that  $\langle \vec{F}_a \rangle = \vec{0}$  so that

$$\frac{\mu}{2} \frac{d}{dt} \langle r^2 \rangle = \langle \epsilon \rangle, \quad (11)$$

In Eq. (11) we will now use the expression given by (9), for  $\langle \epsilon \rangle$ , since the relativistic parameter is significant. We finally get, following the usual steps to compute the mean-square displacement [11]:

$$\langle [r(t)]^2 \rangle = \frac{2}{\mu} \left[ 3kT + mc^2 \frac{K_1\left(\frac{1}{z}\right)}{K_2\left(\frac{1}{z}\right)} \right] t, \quad (12)$$

which reduces to the well-known Einstein-Smoluchowsky equation for the non-relativistic case, if we subtract the rest energy to the internal energy given by Eq. (9). Historically, this equation has been used to establish the value of Boltzmann's constant  $k$  from direct measurement of the mean square displacement in various experimental situations [9]. In figure 2, we plot the next equation in order to compare the relativistic mean square displacement with its non-relativistic counterpart:

$$\left\langle \frac{r^2}{2Dt} \right\rangle = 2 \left( 1 + \frac{1}{3z} \frac{K_1\left(\frac{1}{z}\right)}{K_2\left(\frac{1}{z}\right)} - \frac{1}{3z} \right), \quad (13)$$

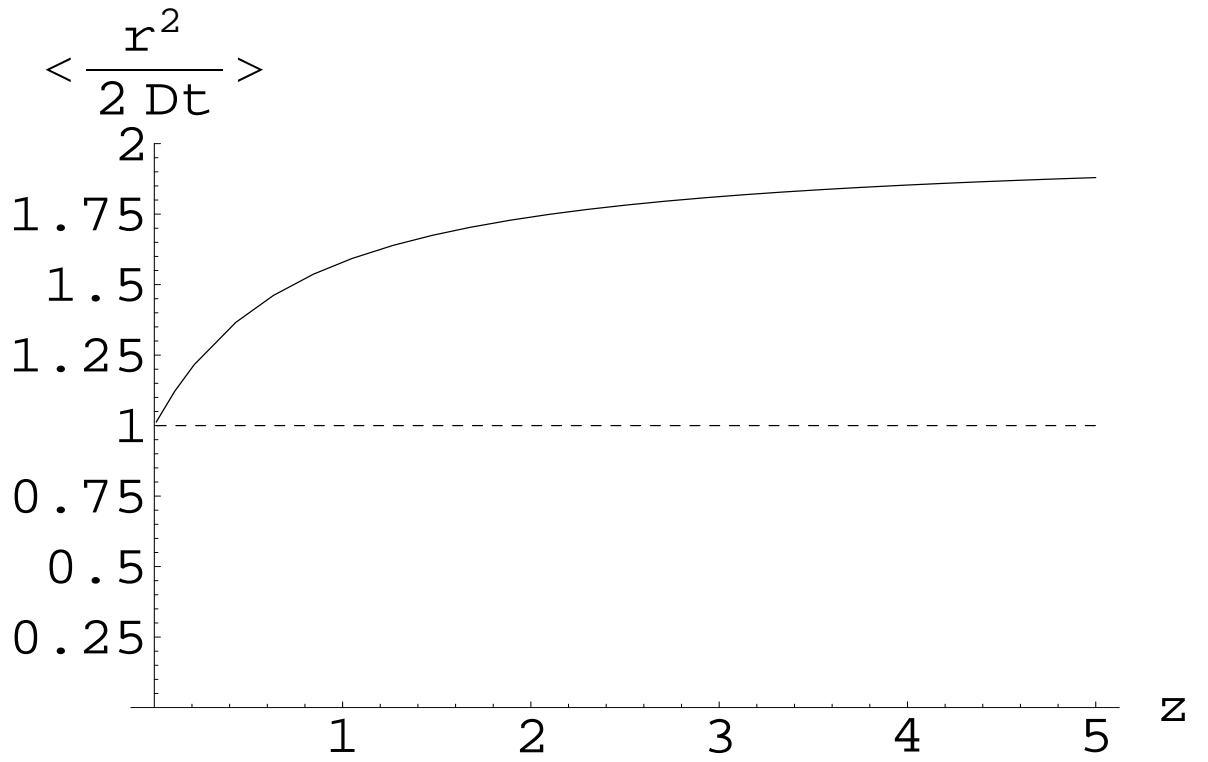
where  $D = \frac{3kT}{\mu}$ . For  $z \ll 1$  we obtain the correct non-relativistic limit

$$\left\langle \frac{r^2}{2Dt} \right\rangle \sim 1. \quad (14)$$

## 4 Final remarks

The framework here presented provides a direct prediction for the value of the mean square displacement of a massive particle exhibiting Brownian motion in a mildly relativistic, simple gas. There is an elementary physical argument to support the approach here proposed. Brownian motion is caused by random collisions of light particles with massive particles. In a certain range of temperatures, the relativistic parameter  $Z = \frac{kT}{Mc^2}$  for the massive particles can be considered

Figure 2: Comparison of the values of the mean square displacement for relativistic and non-relativistic temperatures. For different values of  $z$  the asymptotic behavior of the ratio tends to 1.



negligible, allowing a classical description for their dynamics, while for the thermal bath,  $z \simeq 1$  is significant. Since a non-relativistic distribution predicts an important fraction of particles with supraluminal velocities for large  $z$ , a relativistic treatment describing the properties of the gas is certainly needed. Other calculations of the mean square displacement in this physical situation may require the use of numerical methods or more elaborated approaches [12].

A similar calculation has been performed in reference [12] in the case of a Wiener process in which  $r^2$  is time-dependent for a given value of  $z$ . Work in this direction will be performed in the future.

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